# Evaluation of the Economic Impact of California's Tobacco Control Program: A Dynamic Model Approach--Appendix 1: A theoretical distribution of the tobacco-exposure of cigarette smokers when cigarettes are assumed to be a fixed product.

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> She rang under my feet like an empty Huntley & Palmer biscuit-tin kicked along a gutter; she was nothing so solid in make, and rather less pretty in shape, but I had expended enough hard work on her to make me love her. No influential friend would have served me better. She had given me a chance to come out a bit--to find out what I could do. No, I don't like work. I had rather laze about and think of all the fine things that can be done. I don't like work--no man does--but I like what is in the work,--the chance to find yourself. Your own reality--for yourself, not for others--what no other man can ever know. They can only see the mere show, and never can tell what it really means.
>
> Joseph Conrad,
> Heart of Darkness.

## 1. Introduction

The 25th Surgeon General's report proclaimed that "true scientific understanding of the health effects of tobacco" were achieved in the 20th century" (U.S. Department of Health and Human Services,1989). Major stepping stones in this understanding include Broders' (1920) link between tobacco use and lip cancer; Lombard and Doering's (1928) link between smoking and cancer; and Pearl's (1938) link between smoking and a shorter life span. By 1957, the national Study Group on Smoking and Health (1957) concluded that the relationship between smoking and lung cancer was causal. In a very short time, the Royal College of Physicians(1962) began to extended the adverse of effects of tobacco to a host of other diseases. The analysis reported here is a part of this unfolding of our scientific understanding of the relationship between smoking cigarettes and health. The act of smoking draws tobacco-toxin exposure into the lungs. The more packs-per-day smoked, the more years one smokes, the deeper one inhales, the higher the tars per cigarette the greater the amount of tobacco-exposure deposited in the body. Countervailing this smoker's ingestion process is a biological processes whose purpose is to expel extrinsic objects from the body. These two processes yield a resultant level of tobacco-toxin exposure in a smoker's body at any time. The purpose of this appendix is to derive the theoretical distri bution of these body resident tobacco-exposures for both current and former cigarette smokers, given particular smoking histories, and assuming that cigarettes have been a constant product (fixed level of tars per cigarette). I then present an outline of how knowledge of this distribution will be used to study the effect of smoking on health outcomes.

The tobacco-exposure distribution to be derived here arises from consideration of a stochastic dynamically described accumulation process. Formally, the process is described by a stochas tic differential equation system. Section 2 explicates the two equation accumulation process for current cigarette-smokers. The process is described and a summary of its solution is presented. The *Mathematica* program for the complete current-smoker solution is presented in Appendix 1 to this appendix. Section 3 parallels Section 2; its focus is on the stochastic dynamic tobacco-exposure accumulation process for former cigarette-smokers. Again, the process is described and a summary of its solution is presented. The *Mathematica* program for the complete former-smoker solution is presented in Appendix 2 to this appendix.

The analyses show that resident tobacco-exposures are normally distributed and that they have a heterogeneous variance. Closed form expressions for the expected value and the variance of these distributions are derived. The tobacco-exposure distri bution resulting from the analysis of a current cigarette-smoker is a function of five parameters; the distribution result ing from the analysis of a former-cigarette smoker is a function of one additional parameter. In section 4, I briefly outline how the parameters of these distributions are to be estimated and how the expressions of the moments of the tobacco-exposure distributions will be used to explain the effect of cigarette smoking on diseases caused by smoking and on health status.

Why engage in such an effort? As Lewontin (2003) suggests, the answer might come from consideration about the work such a theoretical distribution would provide? "Science and Simplicity", New York Review of Books, May 1, 2003, pp.39-42). Offering two classes of answers to this work question, he begins with, "Sometimes theoretical structures are nothing but calculat ing devices..." Indeed, that is precisely the principal use that will be made of the tobacco-exposure distribution derived here. As outlined in Section 4, in the chapters to follow, empirical exercises apply the expected value of this distribution to estimate the consequence of smoking on health outcomes. For each age and smoking history, I estimate the effect of smoking on the probability of death. Then, based on the estimated parameters, the distributions of exposure are predicted and used to estimate the effect of a particular smoking history on: (1) the probability of being currently treated for each of two classes of smoking related diseases (SRDs); (2) the distribution of self-reported health status for those not currently treated for SRDs; and on (3) the marginal cost of treatment. Recognizing that all of this has been done many times before, see Max () for a review of studies and for the range of obtained estimates, what can be learned from the effort to be constructed? First, because the full information about an individual's smoking history is not incorporated into existing cross sectionally based estimates of the consequence from smoking, the existing estimates contribute little to understanding the economic consequences of changes in smoking behavior. One of the principal benefits from such an effort is the knowledge gained from replacing the presently employed calculating devices with the calculating device that will be derived here.

In efforts to understand the economic costs of smoking, the usual "calculating device" allocates individuals into smoking history categories with current, former, and never-smoker being the categories most commonly employed. Since there is relatively little variation in the age when smoking is initiated, when an estimated fractional allocation of medical expenditures to smoking is age/gender and smoking status specific, estimates for current-smokers are probably reasonably accurate. However, the age when an individual quits smoking is not part of the existing specifications for former-smokers and the effect of this variation is not deducible from the estimates obtained for former-smokers. One consequence is that existing studies make almost no contribution to understanding the economic benefits arising from smoking cessation programs. The possible complications and the observed averages serve to further confuse. In fact, annual estimates of the level of expenditure or of the smoking attributable fraction for former smokers are often greater than annual estimates for current smokers. If one believes smoking is unambiguously detrimental to health, these findings are only understandable when recognition is made of the fact that smokers quit for different reasons. Some smokers become sick with a smoking related disease. Upon physician's advise, they quit. Their increased medical expenditures are associated with their smoking related disease. While these additional expenditures are appropriately allocatable to smoking, they do not reflect their newly initiated category of "former-smoker." Other smokers have a revelation about the importance of health on their own and their family's well-being and they quit as a means to an end. These individuals may also make greater use of discretionary medical expenditures, but these additional medical expenditures arise from a change in the quitter's demand for health services. These expenditures are not allocatable to tobacco-usage either.

In addition to the lack of a complete description (or of the major dimensions of) an individual's smoking history, another important dimension about smoking history that is omitted from most of the extant specifications focuses on dosage. If one is concerned with the economic benefits associated with smoking cessation programs, as the sponsors of this research are, one would want to be able to estimate the health benefits that accompany a reduction in some of the populations' daily consumption of cigarettes, and/or the health benefits associated with quitting for particularly critical periods of time, such as during the period that a woman is pregnant. Since dosage is not integrated into the current "computational devises", any benefits derived from dosage reduction are not addressable with the existing smoking computational devises.

Programs established to promote cessation in smoking behavior reap benefits when they reduce the smoking attributable phys ical outcomes requiring medical services. The economic evaluation of these programs require being able to estimate the reduction in medical services caused by smoking, given smoking history. To estimate such results is precisely the point of this effort. The theoretical "work" makes feasible estimates of the physical consequences of smoking on a full (perhaps fuller is a better way to put it) statement about an individual's smoking history.

Lewontin's (2003) second category for theory work is that it "...help(s) us "understand" a process whose outcome has been observed but whose dynamical details are not known from experiment or observation." For the topic at hand, Lewontin is discussing science as understood by scientists. There has been a great deal of work of late understanding the dynamic process of smoking induced cellular abnormality development (REF to latest surgeons general report). For example, exposure to a number of things in everyday life, from sunlight to cigarette smoke, can degrade DNA, but our bodies have developed mechanisms to mitigate this damage (Sarah Graham, Scientific American.com, News, September 03, 2003). Livneh and colleagues (Journal of the National Cancer Institute) studied the role of a repair enzyme known as OGG1 in preventing lung cancer. OGG1 deletes DNA parts that have been damaged by oxygen radicals. The theoretical tobacco-exposure distribution derived here neither makes, nor is intended to make any contribution to the scientific under standing of the dynamic biologic process leading to disease at the heart of the material under discussion. Here, the theory offered is merely a metaphor for the biological process.

However, the diminishment in health, the diseases and the deaths caused by smoking are the single most preventable public health hazard. The population of smokers is the group most in need of understanding the true consequences of smoking behavior. Some argue that the negative effects of smoking are common knowledge and smoking is an expression of rational consumer choice (REF). Yet studies show that few have good estimates about the details of these negative effects (REF). It could very well be that biological metaphors effectively transmit the essence of the biological process and contribute toward conveying a general sense of understanding. Computation devises based on believable/understandable metaphors are more likely to lead to believable results. The closer the metaphor's structure is to the true underlying scientific process, the more cred ible the computational device, the more general "understanding" can be derived from the computed knowledge. In addition to the work of understanding, belief serves as a source for judgements. Judges, juries, legislatures, public health bureaucrats, individuals, everyone of us needs to understand the health destructive consequences of smoking. It is toward this understanding that the work of the theory derived here is addressed.

## 2. A description of the tobacco-exposure accumulation process of current cigarette-smokers.

For current-smokers, the level of tobacco-exposure and the time rate of change of this level are denoted $tox_C[t]$ and $tox_C'[t]$, respectively. The first truth about the postulated tobacco-exposure accumulation process (alternatively, read equation describing the process) is an accounting identity that describes the time rate of change of the level of tobacco-exposure in the body of a current-smoker. At each moment t, the change in tobacco-exposure level is simply the difference between the tobacco-exposure ingested through smoking cigarettes and the tobacco-exposure purged from the body through the body's natural process to rid itself of foreign mate rial. It is helpful to think of the moment of time denoted by t as a day in the life of a current-smoker. In such a temporal framework, the ingestion of tobacco-exposure is given by the product of the exposure per pack of cigarettes, denoted by $\delta$, and the number of packs of cigarettes the current-smoker smokes in a day, denoted by p. Thus, at time t, tobacco-exposure ingestion is given by $\delta$ p.

While it is clear that neither $\delta$, nor p have necessarily been constant over the smoking history of any individual, to simplify I assume that both the exposure-per-pack, $\delta$, and the packs-per-day smoked have been constant over an individual's smoking history.

At time t, the rate any current-smoker is able to purge him or herself of tobacco-exposures is denoted $\nu_c[t]$.

Given this notation describing ingestion and purging, the accounting identity describing the time rate of change of the accumulation of tobacco-exposures for a current-smoker at time t is given by equation [2.1],

[2.1] $tox_c'[t] = \delta\ p - \nu_c[t]$.

Factors that affect the purge rate constitute the second half of the tobacco-exposure accumulation process. The second truth (equation) about the exposure accumulation process describes the time rate of change of the tobacco-exposure purge rate. For current smokers, the time rate of change of the purge rate is denoted by $\nu_c'[t]$ . I assume three factors effect the time rate of change of the purge rate. The first factor is aging. Aging causes a decline in all the body's somatic functioning; aging causes the body to be less efficient. More to the point, aging causes a decline in the body's ability to purge itself of exposure, including tobacco-exposures. Accordingly, I assume that the time rate of change in the purge rate declines at a constant rate with aging. This rate is denoted by $\gamma 0$.

Second, I assume that the level of accumulated tobacco-exposures in the body negatively affects the efficiency with which the body expels tobacco-exposures. Increasing tobacco-exposure levels cause a decline in the purge rate. A unit change in accumulated tobacco-exposures causes a $\gamma 1$ decline in the body's purge rate.

Third, I assume that individuals have different somatic reactions to tobacco-exposures. This variability in reaction (think allergic variability) is captured by incorporating a random process into the description of the time rate of change of an individual's purge rate.

The random elements incorporated into this description of the change in the purge rate are instantaneous. I assume that this instantaneous randomness is described by a Wiener process, which is a standardized Brownian motion process. If $\sigma_c$ denotes the standard deviation of a Brownian motion process at time t, and if $d\omega_c$ denotes the time rate of change of a Wiener process at time t, $\sigma_c \, d\omega_c$ denotes the magnitude of the resolution of the instantaneous random shocks occurring at t. Incorporating these three effects results in the description of the time rate of change in the current smoker's purge rate given by equation [2.2],

[2.2] $\nu_c'[t] = -\gamma_0 - \gamma_1 \text{tox}[t] + \sigma_c \, d\omega_c$, with $\gamma_0>0$, $\gamma_1>0$, and $\sigma_c>0$.

There are two initial conditions on this accumulation system. First, the body's tobacco-exposure level when smoking is initiated, $\text{tox}_c[0]$, equals zero. This analysis assumes away second-hand smoke effects. Second, the initial purge rate, $\nu_c[0]$, is an unknown parameter of the problem and denoted $\nu_{c0}$. These initial conditions are described by equation [2.3]

$$[2.3] \quad X0 = \begin{pmatrix} \text{tox}_c[0] \\ \nu_c[0] \end{pmatrix} = \begin{pmatrix} 0 \\ \nu_{c0} \end{pmatrix}.$$

- **A Solution to the stochastic description of the current-smokers tobacco accumulation process**

Here, I present a solution for the dynamic stochastic differential equation system given by equations [2.1] through [2.3]. The solution yields knowledge about the distribution of tobacco-exposures and purge rate implied by the tobacco-exposure accumulation process described above. I begin by representing equations [2.1]-[2.3] in matrix form. For notational purposes, let $dX[t]$ denote a vector describing the first derivatives of the principal variables in the accumulation process of the current-cigarette smoker. Its first element is the time rate of change of the body's tobacco-exposures; its second element is the time rate of change of the purge rate,

[2.4] $$dX[t] = \begin{pmatrix} tox_c'[t] \\ \nu_c'[t] \end{pmatrix}.$$

Let A denote the matrix relating the derivatives of the variables to their magnitudes,

[2.5] $$A = \begin{pmatrix} 0 & -1 \\ -\gamma 1 & 0 \end{pmatrix}.$$

Let H denote a matrix of the constants in the description of the system,

[2.6] $$H = \begin{pmatrix} p\,\delta \\ -\gamma_0 \end{pmatrix}.$$

Let K denote the matrix of constants multiplying the Wiener process for each equation in the system,

[2.7] $$K = \begin{pmatrix} 0 \\ \sigma_c \end{pmatrix}.$$

X0, the initial values of the system, were given by equation [2.3] above. And let $\varpi_t$ indicate a Wiener process at time t.

For a very different framework, Oksendal (2000) presented a solution to a problem with same mathematical structure as the stochastic differential system under analysis here.(Stochastic Differential Equations, An Introduction with Applications, 5th Edition,Springer,pp.64-65). If the exposure accumulation process unfolds between time t0 and time t, X[t], the magnitudes of the accumulated tobacco-exposures and the purge rate, is given by equation [2.8],

$$[2.8] \quad X[t] = \text{MatrixExp}[A\,(t - t_0)].X0 +$$

$$\text{MatrixExp}[A\,(t - t_0)].\text{MatrixExp}[-A\,(t - t_0)].K.\varpi[t] +$$

$$\text{MatrixExp}[A\,(t - t_0)].\int_{t_0}^{t} \text{MatrixExp}[-A\, s].H\, ds +$$

$$\text{MatrixExp}[A\,(t - t_0)].\int_{t_0}^{t} \text{MatrixExp}[-A\, s].A.K.\varpi[s]\, ds.$$

Note that MatrixExp[<arg>] evaluates the power series for the exponential function with ordinary powers replaced by the matrix <arg> (Wolfram, 1996,p.846). Recognize that $t_0 = 0$ and $tox_c[0] = 0$, the closed form solutions for the magnitude of tobacco-exposures and the purge rate, respectively, simplify to equations [2.9a] and [2.9b] (the *Mathematica* derivation of this answer is contained in Appendix 1 to this appendix).

$$[2.9a] tox_c[t] = \left\{\frac{1}{2\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1 + e^{t\sqrt{\gamma 1}}\right)^2 \gamma_0 + \left(-1 + e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta - \nu_{c0})\right)\right)\right\} + \left\{-\left(\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(\omega_t - \omega_0)\right) \Big/ \left(\sqrt{\gamma 1}\right)\right\}$$

$$[2.9b] \nu_c[t] = \left\{\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0 + \sqrt{\gamma 1}\left(p\,\delta + \text{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta + \nu_{c0})\right)\right)\right\} + \left\{\text{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(\omega_t - \omega_0)\right\}$$

The instantaneous random white noise of the Wiener processes integrates over time to a Normally distributed random variable with a heterogeneous variance. In these expressions, the term $(\omega_t - \omega_0)$ represents a Wiener process between time 0 and time t. Wiener processes of duration t have expected values equal to zero, and variances equal to $\sigma^2$t, where $\sigma^2$ in this analysis is given by $\sigma_c^2$.

For expository purposes, I have separated the expressions on the right-hand side (RHS) of equations [2.9a] and [2.9b], with curled brackets, "{}".  The first term (on the RHS) of each of these expressions is the expected value of the magnitude described at time t, the body's tobacco-exposures at time t in equation [2.9a]; the body's purge rate at time t in equation [2.9b].  Each expected value describes its value after a smoking duration of length t, with a dosage of p packs of cigarettes smoked per day.  The second term in each of the above expressions is a random variable, the difference between an individual's true tobacco-exposure level and his expected tobacco-exposure level ([2.9a]; the difference between an individual's true purge rate and his expected purge rate, given the smoking history described by t and p.

Given values for $\gamma 1$, $\gamma 0$, $\delta$, $\nu_{c0}$, and $\sigma_c^2$, the parameters of the tobacco-exposure accumulation system, tobacco-exposure and purge-rate levels are linear transformations of a normal random variable. Accordingly, the tobacco-exposure and purge rate levels are normally distributed.  Since the variance of a constant times a random variable is equal to the product of the square of the constant and the variance of the random variable, the variance of the tobacco-exposure level, denoted by $\sigma^2{}_{toxc}[t]$, is given by equation [2.10a],

$$[2.10a] \quad \sigma^2{}_{toxc}[t] = \left\{ \frac{t \, \text{Sinh}\left[t \sqrt{\gamma 1}\right]^2 \sigma_c^2}{\gamma 1} \right\},$$

and the variance of the purge rate, denoted by $\sigma^2{}_{\nu' c}[t]$, is given by equation [2.10b],

$$[2.10b] \quad \sigma^2{}_{\nu c}[t] = \left\{ t \, \text{Cosh}\left[t \sqrt{\gamma 1}\right]^2 \sigma_c^2 \right\}.$$

Sinh[<arg>] and Cosh[<arg>] are respectively the hyperbolic sine and hyperbolic cosine of the argument <arg> (Wolfram, 1996, p.731).

**3. A description of the tobacco-exposure accumulation process of former cigarette-smokers.**

The subscript c was used in the presentation of Section 2 above to denote the current smoker's model. Here, the subscript f will be used to denote the former-smokers model. Fundamentally, the model describing a former-smoker is based on the same mathematical structure as the model describing a current smoker. However two variable values are different, one parameter is allowed to be different, and the initial values of the magnitudes of the system, an individual's level of tobacco-exposures and his purge rate are different. We begin with the changes in the variable values. First, time for the former smoker is a count of the time of abstention, not time of smoking. The variable u denotes the length of time of abstinence. The time when a current-smoker quit is denoted by te (think e for end). When a current-smoker transitions to a former-smoker, t=te, u=0. For a former-smoker, the current-smoker's model applies for the period in which he smoked, that is, from t=0 to t=te. Take two individuals who are identical with the single exception that one stopped smoking at time te, $0 < te < t$. The value t for the current smoker equals te + u for the former smoker. The second variable value that changes in the former-smoker's model is the dosage measure, the packs-per-day smoked. During the period of abstention zero packs-per-day are consumed. Thus, the ingestion described in the time rate of change of the tobacco-exposure accounting identity during the former-smoker's abstention period has a value of zero. Since the body's purging process continues to operate, similar to Section 2 above, equation [3.1], describes the time rate of change in the tobacco-exposure level without ingestion, but with a continuing purging process. Recall, time, denoted by u, measures the duration of the abstention period,

$$[3.1] \quad tox_f'[u] = -\nu_f[u].$$

The biological metaphor does not change when the individual changes smoking status. The time rate of change in the former-smoker's purge-rate, $\nu_f'[u]$, remains affected by the same three factors that affect the time rate of change in the purge-rate of a current-smoker. However, I allow for the possibility that the instantaneous random process during the period of abstinence can have a different standard deviation than the standard deviation operating during the period of cigarette consumption. Equation [3.2] specifies the time rate of change of the purge rate of a former smoker,

[3.2] $\nu_f'[u] = -\gamma_0 - \gamma_1 \, tox_f[u] + \sigma_f \, d\omega_u$, with $\gamma_0>0$, $\gamma_1>0$, and $\sigma_f>0$.

The third difference between the current and former smoker's models is the description of the values of the magnitudes of the variables of the system when the system begins; the initial conditions of the system. The initial condition for a current smoker was the description of the stochastic differential system at time t equal to zero. The initial condition for a former-smoker is the description of the stochastic differential system at time t equal to te (u=0). When a current-smoker transitions to a former smoker, his expected level of tobacco-exposures and his expected purge-rate equals its value as a current smoker, see equations [2.9a] and [2.9b] above. Equation [3.3] describes the initial conditions at the moment a current-smoker transitions to a former smoker,

[3.3] X0 = $tox_f[u=0]$ =

$$\left\{\frac{1}{2\gamma 1}\left(e^{-te\sqrt{\gamma 1}}\left(\left(-1+e^{te\sqrt{\gamma 1}}\right)^2\gamma_0 + \left(-1+e^{2\,te\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)\right\} + \left\{-\frac{1}{\sqrt{\gamma 1}}\left(\text{Sinh}\left[te\sqrt{\gamma 1}\right]\sigma_c\,(\omega_t-\omega_0)\right)\right\},$$

$$\nu_f[u=0] \quad \left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[te\sqrt{\gamma 1}\right]\gamma_0 + \sqrt{\gamma 1}\left(p\,\delta + \mathrm{Cosh}\left[te\sqrt{\gamma 1}\right](-p\,\delta + \nu_{c0})\right)\right)\right\} + \left\{\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]\sigma_c\,(\omega_t - \omega_0)\right\}.$$

Similar to equation [2.4], the vector of first derivatives of the variables of the system, evaluated at time u, is denoted by $\mathbb{d}X[u]$. It is given by equation [3.4],

$$[3.4] \quad \mathbb{d}X[u] = \begin{pmatrix} tox_f'[u] \\ \nu_f'[u] \end{pmatrix}.$$

As in the description of a current-smoker, the matrix A is the matrix of coefficients relating the derivatives of the variables to their magnitudes, the matrix H is the matrix of constants relating the derivatives to their magnitudes. These matrices are the same in both systems. The matrix K is the matrix of constants multiplying the Wiener processes associated with each of the two equations in the system. The only difference between the matrix K in a current-smoker's model and the matrix K in a former-smoker's model is the subscript on the stan dard deviation of the white noise process. Equation [3.5] is the appropriate representation of the K matrix for a former-smoker's model.

$$[3.5] \quad K = \begin{pmatrix} 0 \\ \sigma_f \end{pmatrix}.$$

The essence of the solution to this system is again given by equation [2.7]. Equations [3.6a] and [3.6b] present the simplified closed form solutions for the magnitude of the tobacco-exposures and the purge-rate, respectively, for a former-smoker. The full *Mathematica* derivation of this answer is contained in Appendix 2 to this appendix).

$$[3.6a] \quad tox_f[u,te] =$$

$$\left\{\frac{1}{2\gamma 1}\left(e^{-(te+u)\sqrt{\gamma 1}}\left(-2\,e^{(te+u)\sqrt{\gamma 1}}\,p\sqrt{\gamma 1}\,\delta\,\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]+\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)\right\}+$$

$$\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(\omega_u-\omega_0)\right)\right\}+\left\{\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]\right\}$$

$$+\left\{-\frac{1}{\sqrt{\gamma 1}}\left(\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\nu c}[te]\right)\right\}.$$

[3.6b] $\nu_f[u,te] =$

$$\left\{\frac{1}{\sqrt{\gamma 1}}\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]-\mathrm{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\,\mathrm{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right\}+$$

$$\left\{\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(\omega_u-\omega_0)\right\}+$$

$$\left\{-\sqrt{\gamma 1}\,\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]\right\}$$

$$\left\{\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\nu c}[te]\right\}.$$

Completely analogous to the solution equations in Section 2 above, I have separated the expressions on the RHS of equations [3.6a] and [3.6b] with curled brackets, "{}". The first expression in each equation is the expected value of the respective magnitude, given estimated parameter values and values for how long the former-smoker smoked, te, and how long a respondent abstained from smoking, u. The second expressions are the random errors resulting from the Wiener process for a former-smoker. The third expressions are the consequence of the deviation between the true tobacco-exposure level and the expected tobacco-exposure level when the current-smoker transitions to a former-smoker. This deviation was then acted upon by the former-smoker's stochastic dynamic tobacco-exposure accumulation process. The fourth (last) expressions are the consequence of the deviation between the true purge-rate level and the expected purge-rate level, again when the current-smoker transitions to a former-smoker. This deviation, again, was subsequently acted upon by the former-smoker's stochastic dynamic tobacco-exposure accumulation process.

Given values for $\gamma 1$, $\gamma 0$, $\delta$, $\nu_{c0}$, $\sigma_c^2$, and $\sigma_f^2$, the parameters of a former-smoker's tobacco-exposure accumulation system, the tobacco-exposure and purge-rate levels are linear transformations of normal random variables. Accordingly, these tobacco-exposure and purge-rate levels are normally distributed. The variances of these respective levels are denoted $\sigma^2{}_{toxf}[u, te]$, and $\sigma^2{}_{toxf}[u]$. Again, for identification purposes, the right-hand terms of equations [3.7a] and [3.7b], expressions for these variances, are separated with curly brackets. The first term on the RHS of equations [3.7a] and [3.7b], respectively, is the variance of the Wiener process acting on the exposure level and purge rate, respectively. The second terms are the variance induced by the initial difference between the true and expected tobacco-exposure levels. The third terms are the variance induced by the initial difference between the true and expected purge rate,

[3.7a] $\sigma^2{}_{toxf}[u, te] =$

$$\left\{\frac{u\,\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2\sigma_f^2}{\gamma 1}\right\} + \left\{\frac{1}{\gamma 1}\left(te\,\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2\mathrm{Sinh}\left[te\sqrt{\gamma 1}\right]^2\sigma_c^2\right)\right\} + \left\{\frac{1}{\gamma 1}\left(te\,\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]^2\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2\sigma_c^2\right)\right\},$$

[3.7b] $\sigma^2{}_{\nu f}[u, te] =$

$$\left\{u\,\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2\sigma_f^2\right\} + \left\{te\,\mathrm{Sinh}\left[te\sqrt{\gamma 1}\right]^2\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2\sigma_c^2\right\} + \left\{te\,\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]^2\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2\sigma_c^2\right\}.$$

**4. Towards achieving a calculation device based on these theo-**

**retic tobacco-exposure distributions.**

**Incorporating the theoretic tobacco-exposure distribution into behavioral specifications**

All of the empirical analyses to follow in this study estimate the effect of smoking on health and cost related outcomes. The derived distribution of accumulated tobacco-exposures is used as a tobacco-exposure index to portray a history of smoking behavior. In all of the analyses, the propensity for the occurrence of the health or cost outcome under analysis is set equal to parameter weighted measures of the observation's age and his accumulated tobacco-exposure level. The coefficient weighted expected level of tobacco-exposures are a RHS variable in the specification of the expected propensity under analysis; the coefficient weighted difference between the true and expected level of tobacco-exposures is, in effect, part of the random error of the model.

The feasibility of this plan requires (1) recognition that at least a part of the random error term in each model has a normal distribution; and (2) estimates of the parameters of the exposure accumulation model, so that the expected exposure level and its variance can be estimated and incorporated into the analyses. In the next appendix to follow a survival model is proposed that has the appropriate error distributions. I call it a Normally Distributed Survival Model. The model is difficult to estimate and starting values are quite critical. In Chapter Three I suggest methods that one can employ to achieve starting values to use with estimating the proposed Probit Survival Model. Chapter Four presents a simple use of the Probit Survival Model. The propensity to die is estimated for two classes of never-smokers, those who have had no higher education and those who have had some higher education. Higher education is being used as a proxy for social class, which is a proxy for access to health care services. Chapter Five presents estimates of the propensity to die for current smokers based on the Probit Survival Model specification. The estimates for the never-smokers are used to characterize the effect of age and health care access. The model estimates the parameters of a simplified version of the current-smokers model. Chapter Six presents estimates of the propensity to die for former-smokers, again based on the Probit Survival Model and again with estimates for the never-smokers characterizing the effect of age and health care access. The somewhat simplified version of the current smokers model is extended to represent the former smokers.

With the obtained tobacco-exposure parameter estimates, given an individual's smoking history, it is possible to calculate expected levels of tobacco-exposures and standard deviations in those levels. The next three analyses are based on these calculated values. In Chapter Seven, I estimate the probability of being currently treated for a class of smoking related diseases (lung cancer, esophageal cancer, and chronic obstructive pulmonary disease) that compared to never-smokers have a high relative risk due to smoking. In Chapter Eight the exercise is repeated for the class of smoking related diseases (all of the remaining smoking related diseases) that have a relatively low relative risk due to smoking. In Chapter Nine, for a sample that is not currently treated for a smoking related disease I estimate the effect of smoking on self-reported poor health status.

These analyses allow comparisons between a wider selection of smoking histories than is usually made. They also allow correction for sample selection due to death, obtaining a more accurate (and unbiased) estimate of the relative risk of current treatment statuses induced by various intensities of smoking behavior. The set of analyses also allows correction for the additional contributor to sample selection, current treatment status, to yield an unbiased estimate of the effect of smoking on the probability distribution of self-reported poor health status.

Chapter Ten estimates a medical expenditure model for people who are not currently treated for a smoking related disease. Again, smoking status is based on the distribution of tobacco-exposures. The specification is able to differentiate among the tobacco-exposure effect of the demand for medical services and any change in demand for medical services that accompany a shift in smoking status from current to former.

Chapter Eleven uses all of the derived models to estimate the expected deaths, the distribution of expected smoking related disease treatment, and the distribution of the self-reported health status of California's population for a considerable period into the future. Chapter Eleven estimates the smoking prevalence rates and quit rates that California might have had, if its Tobacco Prevention Program had not existed, and Chapter Twelve compares the simulations performed on California's population to the simulations performed on California's population without its tobacco prevention program to estimate the economic and physical benefits from the program over the decade of the nineteen nineties.

# Appendix 1.1: Solution for Current Smoker

The model to be solved, representing the generalized tobacco-exposure accumulation model for the current-smoker is as follows:

$tox_c'[t] = \delta\, p - \nu_c[t]$,
$\nu_c'[t] = -\gamma_0 - \gamma_1\, tox[t] + \sigma_c\, d\omega_t$,

where: tox[t]= accumulated level of tobacco originating exposures in the body;
$\nu_c[t]$ = body purge rate;
$\gamma_0$= drift rate in the body's purging ability due to aging;
$\gamma_1$ = drift rate in the body's purging ability due to exposure accumulation;
$\sigma_c$= standard deviation of the Wiener stochastic process;
$d\omega_t$= Wiener process at time t;
$\delta$ = density of exposures per pack of cigarettes;
p = packs of cigarettes smoked per day.

The vector of first derivatives evaluated at time t is dX[t],

```
dX[t] = {{toxc'[t]}, {νc'[t]}}
```

$\{\{tox_c'[t]\}, \{\nu_c'[t]\}\}$

```
MatrixForm[%]
```

$$\begin{pmatrix} tox_c'[t] \\ \nu_c'[t] \end{pmatrix}$$

A is the matrix relating the derivatives of the variables in the model to the variables in the model,

```
A = {{0, -1}, {-γ1, 0}}
```

$\{\{0, -1\}, \{-\gamma 1, 0\}\}$

```
MatrixForm[%]
```

$$\begin{pmatrix} 0 & -1 \\ -\gamma 1 & 0 \end{pmatrix}$$

H is a matrix of constants,

**H = {{δ p}, {-γ0}}**

$\{\{p\,\delta\}, \{-\gamma_0\}\}$

K is a matrix of constants multiplying the Wiener processes associated with each equation,

**K = {{0}, {σc }}**

$\{\{0\}, \{\sigma_c\}\}$

X0 denotes the matrix of starting values,

**X0 = {{toxc0}, {νc0}}**

$\{\{tox_{c0}\}, \{\nu_{c0}\}\}$

W indicates the Wiener process (Standardized Brownian Motion)

**W = {{ω}}**

$\{\{\omega\}\}$

The next four operations are the four parts of the solution.

**MatrixExp[A (t - t0)].X0**

$$\left\{\left\{\frac{1}{2} e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1 + e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) tox_{c0} - \frac{e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1 + e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \nu_{c0}}{2\sqrt{\gamma 1}}\right\}, \left\{-\frac{1}{2} e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1 + e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \sqrt{\gamma 1}\, tox_{c0} + \frac{1}{2} e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1 + e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \nu_{c0}\right\}\right\}$$

**MatrixExp[A ( t - t0)]. MatrixExp[-A ( t - t0)].K.W**

$$\left\{\{0\},\left\{\left(-\frac{1}{4}e^{-2\sqrt{\gamma 1}(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)^2+\frac{1}{4}e^{-2\sqrt{\gamma 1}(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)^2\right)\omega\,\sigma_c\right\}\right\}$$

**MatrixExp[A (t - t0)].**
**Integrate[MatrixExp[-A s].H , {s, t0, t}]**

$$\Bigg\{\Bigg\{-\frac{1}{2\sqrt{\gamma 1}}\Bigg(e^{-\sqrt{\gamma 1}(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)\Bigg(p\,\delta\left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]-\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)+\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\gamma_0\Bigg)\Bigg)+\frac{1}{\gamma 1}\Bigg(e^{-\sqrt{\gamma 1}(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}(t-t_0)\right]\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}(t+t_0)\right]-\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}(t+t_0)\right]\gamma_0\right)\Bigg)\Bigg\},$$

$$\Bigg\{\frac{1}{2}e^{-\sqrt{\gamma 1}(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)\Bigg(p\,\delta\left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]-\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)+\frac{\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\gamma_0}{\sqrt{\gamma 1}}\Bigg)-\frac{1}{\sqrt{\gamma 1}}\Bigg(e^{-\sqrt{\gamma 1}(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}(t-t_0)}\right)\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}(t-t_0)\right]\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}(t+t_0)\right]-\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}(t+t_0)\right]\gamma_0\right)\Bigg)\Bigg\}\Bigg\}$$

**MatrixExp[A (t − t0)].
Integrate[MatrixExp[−A s] . A . K . W, {s, t0, t}]**

$$\left\{\left\{-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\right)+\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\right)\right\},\right.$$

$$\left.\left\{\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c-\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\right\}\right\}$$

The solution of the magnitudes is the sum of the four parts given directly above,

**X[t] = %9 + %10 + %11 + %12**

$$\left\{\left\{\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)tox_{c0}-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\left(p\,\delta\left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]-\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)+\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\gamma_0\right)\right)+\right.\right.$$

$$\frac{1}{\gamma 1}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t-t_0)\right]\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]-\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]\gamma_0\right)\right)-$$

$$\frac{e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)v_{c0}}{2\sqrt{\gamma 1}}-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\right)+$$

$$\left.\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\omega\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\right)\right\},$$

$$\Bigg\{-\frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \sqrt{\gamma 1}\; tox_{c0} + \frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \left(p\,\delta \left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]-\mathrm{Cosh}\left[\sqrt{\gamma 1}\; t_0\right]\right) + \frac{\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\; t_0\right]\right)\gamma_0}{\sqrt{\gamma 1}}\right) - \frac{1}{\sqrt{\gamma 1}} \left(e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t-t_0)\right] \left(p\sqrt{\gamma 1}\;\delta\,\mathrm{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]-\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]\gamma_0\right)\right) + \frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \nu_{c0} + \left(-\frac{1}{4}\, e^{-2\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)^2 + \frac{1}{4}\, e^{-2\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)^2\right) \omega\,\sigma_c + \frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \omega \left(-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Cosh}\left[\sqrt{\gamma 1}\; t_0\right]\right)\sigma_c - \frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \omega \left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\; t_0\right]\right)\sigma_c\Bigg\}\Bigg\}$$

At time t, $\omega$ would denote the difference between the Wiener process at t and the Wiener process at 0

**%13 /. $\omega \to (W_t - W_0)$**

$$\Bigg\{\Bigg\{\frac{1}{2}\, e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) tox_{c0} - \frac{1}{2\sqrt{\gamma 1}} \left(e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \left(p\,\delta \left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]-\mathrm{Cosh}\left[\sqrt{\gamma 1}\; t_0\right]\right) + \frac{1}{\sqrt{\gamma 1}} \left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]+\mathrm{Sinh}\left[\sqrt{\gamma 1}\; t_0\right]\right)\gamma_0\right)\right) + \frac{1}{\gamma 1} \left(e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t-t_0)\right] \left(p\sqrt{\gamma 1}\;\delta\,\mathrm{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]-\mathrm{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]\gamma_0\right)\right) - \frac{e^{-\sqrt{\gamma 1}\,(t-t_0)} \left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right) \nu_{c0}}{2\sqrt{\gamma 1}} -$$

$$\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\right.$$
$$\left.\left(-\text{Cosh}\left[t\sqrt{\gamma 1}\right]+\text{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right)+$$
$$\frac{1}{2\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]+\text{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\right.$$
$$\left.\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right)\Big\},$$
$$\Big\{-\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\sqrt{\gamma 1}\,\text{tox}_{c0}+$$
$$\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\left(p\,\delta\left(\text{Cosh}\left[t\sqrt{\gamma 1}\right]-\text{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)+\right.$$
$$\left.\frac{\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]+\text{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\gamma_0}{\sqrt{\gamma 1}}\right)-$$
$$\frac{1}{\sqrt{\gamma 1}}\left(e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\text{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t-t_0)\right]\right.$$
$$\left.\left(p\sqrt{\gamma 1}\,\delta\,\text{Cosh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]-\text{Sinh}\left[\frac{1}{2}\sqrt{\gamma 1}\,(t+t_0)\right]\gamma_0\right)\right)+$$
$$\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\nu_{c0}+$$
$$\left(-\frac{1}{4}e^{-2\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)^2+\right.$$
$$\left.\frac{1}{4}e^{-2\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)^2\right)\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)+$$
$$\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)\left(-\text{Cosh}\left[t\sqrt{\gamma 1}\right]+\text{Cosh}\left[\sqrt{\gamma 1}\,t_0\right]\right)$$
$$\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)-\frac{1}{2}e^{-\sqrt{\gamma 1}\,(t-t_0)}\left(-1+e^{2\sqrt{\gamma 1}\,(t-t_0)}\right)$$
$$\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]+\text{Sinh}\left[\sqrt{\gamma 1}\,t_0\right]\right)\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\Big\}\Big\}$$

Setting the starting values for $tox_c$ and t equal to their known values, the solution equations become:

**{{tox_c[t]}, {ν_c[t]}} = %14 /. {tox_c0 → 0, t_0 → 0}**

$$\left\{\left\{\frac{1}{\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)\mathrm{Sinh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]-\mathrm{Sinh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]\gamma_0\right)\right)-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)\left(p\,\delta\left(-1+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\right)-\frac{\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0}{\sqrt{\gamma 1}}\right)\right)-\frac{e^{-t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)\nu_{c0}}{2\sqrt{\gamma 1}}-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)\left(1-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\right)\sigma_c\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_t\right)\right)-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_t\right)\right)\right\},\right.$$

$$\left.\left\{-\frac{1}{\sqrt{\gamma 1}}\left(e^{-t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)\mathrm{Sinh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]\left(p\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]-\mathrm{Sinh}\left[\frac{t\sqrt{\gamma 1}}{2}\right]\gamma_0\right)\right)+\frac{1}{2}e^{-t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)\left(p\,\delta\left(-1+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\right)-\frac{\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0}{\sqrt{\gamma 1}}\right)+\frac{1}{2}e^{-t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)\nu_{c0}+\left(-\frac{1}{4}e^{-2t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)^2+\frac{1}{4}e^{-2t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)^2\right)\sigma_c\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_t\right)+\frac{1}{2}e^{-t\sqrt{\gamma 1}}\left(1+e^{2t\sqrt{\gamma 1}}\right)\left(1-\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\right)\sigma_c\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_t\right)+\frac{1}{2}e^{-t\sqrt{\gamma 1}}\left(-1+e^{2t\sqrt{\gamma 1}}\right)\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_t\right)\right\}\right\}$$

**FullSimplify[%15]**

$$\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,\left(p\,\delta-\nu_{c0}+\sigma_c\,(\{\{\omega\}\}_0-\{\{\omega\}\}_t)\right)\right)\right)\right\},\right.$$
$$\left.\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\left(-p\,\delta+\nu_{c0}+\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right)\right)\right)\right\}\right\}$$

Note that in the expression for $tox_c[t]$the Wiener process is going backward and needs to be changed. First identify the coefficient on the Wiener process. What follows is the expected value of $tox_c[t]$,

**%16[[1]] /. ({{ω}}₀ - {{ω}}ₜ) → 0**

$$\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)\right\}$$

The difference between the whole expression for $tox_c[t]$ and the expected value of $tox_c[t]$ is the random error in the expression for $tox_c[t]$

**%16[[1]] - %17**

$$\left\{-\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)+\right.$$
$$\left.\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,\left(p\,\delta-\nu_{c0}+\sigma_c\,(\{\{\omega\}\}_0-\{\{\omega\}\}_t)\right)\right)\right)\right\}$$

**FullSimplify[%]**

$$\left\{\frac{\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(\{\{\omega\}\}_0-\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

Multiply this term by -1 and change the signs in the Wiener processes.

**-1 %19**

$$\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(\{\{\omega\}\}_0-\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

$$\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

$$\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

Accordingly, ( from %17 and %19), the following is the correct statement for $\text{tox}_c[t]$, (see Out[26] below),

$$\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)\right\}+$$

$$\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

$$\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\}$$

Returning to the expression for $\nu_c[t]$, its expected value is:

**%16[[2]] /. ({{ω}}$_t$ - {{ω}}$_0$) → 0**

$$\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\text{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)\right\}$$

The difference between the whole expression for $\nu_c[t]$ and the expected value of $\nu_c[t]$ is the random error in the expres sion for $\nu_c[t]$,

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%16[[2]] - %23
```

$$\left\{-\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0}+\sigma_c(-\{\{\omega\}\}_0+\{\{\omega\}\}_t))\right)\right)\right\}$$

```
FullSimplify[%]
```

$$\left\{\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right\}$$

The expression for the solution to the problem is as follows:

```
{%22, %23 + %25}
```

$$\left\{\left\{\frac{1}{2\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)-\frac{\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\},\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right\}\right\}$$

We can express this solution as the sum of its expected values and its error. The expected value results from setting the random error equal to zero. Taking the result from above, (where in the statement to follow the first parts are the expected values and the second parts are the error terms,

$$\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[t]\right\},\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\text{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu_c}[t]\right\}\right\}$$

— *General::spell1 : Possible spelling error: new symbol name "toxc" is similar to existing symbol "tox". More…*

$$\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[t]\right\},\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\text{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu_c}[t]\right\}\right\}$$

The random errors are given by

$$\left\{\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\},\left\{\text{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right\}\right\}$$

$$\left\{\left\{-\frac{\text{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)}{\sqrt{\gamma 1}}\right\},\left\{\text{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_t)\right\}\right\}$$

The following expressions (Out[27] evaluated at t->te describe the starting value for $tox_f[0]$, $\nu_f[0]$, in equation [1.14]

$$\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2\,t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\ (p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[t]\right\},\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu_c}[t]\right\}\right\}$$

$$\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2\,t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\ (p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[t]\right\},\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu_c}[t]\right\}\right\}$$

The variances of $tox_c[t]$ and $\nu_c[t]$ are the variance of these error terms. This expression is the variance squared times t.

$$\left\{\left\{\left(-\frac{1}{\sqrt{\gamma 1}}\left(\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,\mathrm{Sqrt}[t]\right)\right)^2\right\},\left\{\left(\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]\sigma_c\,\mathrm{Sqrt}[t]\right)^2\right\}\right\}$$

$$\left\{\left\{\frac{t\,\mathrm{Sinh}\left[t\sqrt{\gamma 1}\right]^2\sigma_c^2}{\gamma 1}\right\},\left\{t\,\mathrm{Cosh}\left[t\sqrt{\gamma 1}\right]^2\sigma_c^2\right\}\right\}$$

This is a check to insure that the expected value of $tox_{c0} = 0$.

$$\left\{\frac{1}{2\,\gamma 1}\left(e^{-t\sqrt{\gamma 1}}\left(\left(-1+e^{t\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2\,t\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\ (p\,\delta-\nu_{c0})\right)\right)\right\}\ /.\ t\to 0$$

$$\{0\}$$

**Appendix 1.2:Solution for a Former-Smoker**

The model to be solved represents the generalized tobacco-exposure accumulation model for a former smoker is as follows:

$tox_f'[u] = -\nu_f[u]$,

$\nu_f'[u] = -\gamma_0 - \gamma_1 tox[u] + \sigma_f d\omega_u$,

where:

$\nu_f[u]$ = body purge rate for former smokers at time u into abstinence;

$tox_f[u]$= the accumulated level of tobacco originating exposures in the body at time u;

$\gamma_0$= drift rate in the body's purging ability due to aging;

$\gamma_1$ = drift rate in the body's purging ability due to exposure accumulation;

$\sigma_f$= standard deviation of the Wiener stochastic process;

$d\omega_u$= Wiener process at time u;

$\delta$ = density of exposures per pack;

p = packs of cigarettes smoked per day.

The vector of first derivatives evaluated at time u is $dX[u]$

**dX[u] = {{tox_f' [u]}, {ν_f' [u]}}**

$\{\{tox_f'[u]\}, \{\nu_f'[u]\}\}$

**MatrixForm[%]**

$$\begin{pmatrix} tox_f'[u] \\ \nu_f'[u] \end{pmatrix}$$

A is the matrix relating the variable to its derivative.

```
A = {{0, -1}, {-γ1, 0}}
```

$\{\{0, -1\}, \{-\gamma 1, 0\}\}$

```
MatrixForm[%]
```

$$\begin{pmatrix} 0 & -1 \\ -\gamma 1 & 0 \end{pmatrix}$$

H is a matrix of constants relating the derivatives to the magnitudes of the variables.magnitudes

```
H = {{ 0}, {-γ0}}
```

$\{\{0\}, \{-\gamma_0\}\}$

K is a matrix of constants multiplying the Wiener processes associated with each equation.

```
K = {{0}, {σf }}
```

$\{\{0\}, \{\sigma_f\}\}$

X0 denotes the matrix of starting values.

```
X0 = {{toxf0}, {νf0}}
```

$\{\{tox_{f0}\}, \{\nu_{f0}\}\}$

W indicates the Wiener process (Brownian Motion).

```
W = {{ω}}
```

$\{\{\omega\}\}$

The next four operations are the four parts of the solution

**MatrixExp[A (u − u$_0$)].X0**

$$\left\{\left\{\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\text{tox}_{f0}+\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\nu_{f0}\right\},\left\{\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}\right)\text{tox}_{f0}+\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\nu_{f0}\right\}\right\}$$

**MatrixExp[A ( u − u$_0$)]. MatrixExp[−A ( u − u$_0$)].K.W**

$$\left\{\left\{\left(\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)+\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\left(-\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}+\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\right)\omega\,\sigma_f\right\},\left\{\left(\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\right)^2+\left(-\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}+\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}\right)\right)\omega\,\sigma_f\right\}\right\}$$

```
MatrixExp[A (u - u0)].
 Integrate[MatrixExp[-A s].H , {s, u0, u}]
```

$$\left\{\left\{-\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\,\gamma 1}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\,\gamma 1}\right)\gamma 0-\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)\gamma 0\right\},\left\{-\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)\gamma 0-\left(\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\,\gamma 1}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\,\gamma 1}\right)\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}-\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}\right)\gamma 0\right\}\right\}$$

```
MatrixExp[A (u - u0)].
 Integrate[MatrixExp[-A s] . A . K . W, {s, u0, u}]
```

$$\left\{\left\{\left(-\frac{1}{2} e^{-u\sqrt{\gamma1}}\left(1+e^{2u\sqrt{\gamma1}}\right)+\frac{1}{2} e^{-\sqrt{\gamma1}\,u_0}\left(1+e^{2\sqrt{\gamma1}\,u_0}\right)\right)\left(\frac{e^{-\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}-\frac{e^{\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}\right)\omega\,\sigma_f+\left(\frac{1}{2} e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2} e^{\sqrt{\gamma1}\,(u-u_0)}\right)\left(-\frac{e^{-u\sqrt{\gamma1}}\left(-1+e^{2u\sqrt{\gamma1}}\right)}{2\sqrt{\gamma1}}+\frac{e^{-\sqrt{\gamma1}\,u_0}\left(-1+e^{2\sqrt{\gamma1}\,u_0}\right)}{2\sqrt{\gamma1}}\right)\omega\,\sigma_f\right\},\right.$$

$$\left.\left\{\left(\frac{1}{2} e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2} e^{\sqrt{\gamma1}\,(u-u_0)}\right)\left(-\frac{1}{2} e^{-u\sqrt{\gamma1}}\left(1+e^{2u\sqrt{\gamma1}}\right)+\frac{1}{2} e^{-\sqrt{\gamma1}\,u_0}\left(1+e^{2\sqrt{\gamma1}\,u_0}\right)\right)\omega\,\sigma_f+\left(-\frac{e^{-u\sqrt{\gamma1}}\left(-1+e^{2u\sqrt{\gamma1}}\right)}{2\sqrt{\gamma1}}+\frac{e^{-\sqrt{\gamma1}\,u_0}\left(-1+e^{2\sqrt{\gamma1}\,u_0}\right)}{2\sqrt{\gamma1}}\right)\left(\frac{1}{2} e^{-\sqrt{\gamma1}\,(u-u_0)}\sqrt{\gamma1}-\frac{1}{2} e^{\sqrt{\gamma1}\,(u-u_0)}\sqrt{\gamma1}\right)\omega\,\sigma_f\right\}\right\}$$

The solution of the magnitudes is the sum of the four parts given directly above

```
X[t] = %9 + %10 + %11 + %12
```

$$\left\{\left\{\left(\frac{1}{2} e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2} e^{\sqrt{\gamma1}\,(u-u_0)}\right)tox_{f0}-\left(\frac{1}{2} e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2} e^{\sqrt{\gamma1}\,(u-u_0)}\right)\right.\right.$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\gamma 1}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\gamma 1}\right)\gamma_0 -$$

$$\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)\gamma_0 +$$

$$\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\nu_{f0} +$$

$$\left(\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\right)\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\right.\right.$$

$$\left.\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)+\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\right)$$

$$\left.\left(-\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}+\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\right)\omega\,\sigma_f +$$

$$\left(-\frac{1}{2}e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)+\frac{1}{2}e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)\right)$$

$$\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}-\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\omega\,\sigma_f +$$

$$\left(\frac{1}{2}e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma 1}\,(u-u_0)}\right)$$

$$\left(-\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}+\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)$$

$$\omega\,\sigma_f\Big\},$$

$$
\begin{aligned}
&\Big\{\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}\Big)\,tox_{f0}-\\
&\quad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\Big)\\
&\quad\left(\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)\gamma_0-\\
&\quad\left(\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\,\gamma 1}-\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\,\gamma 1}\right)\\
&\quad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}\Big)\,\gamma_0+\\
&\quad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\Big)\,\nu_{f0}+\\
&\quad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\Big)\\
&\quad\Big(-\frac{1}{2}\,e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)+\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,u_0}\left(1+e^{2\sqrt{\gamma 1}\,u_0}\right)\Big)\\
&\quad\omega\,\sigma_f+\Bigg(\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}+\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\Big)^2+\\
&\qquad\left(-\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}+\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\\
&\qquad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}\Big)\Bigg)\,\omega\,\sigma_f+\\
&\quad\left(-\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}+\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)\\
&\quad\Big(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\,\sqrt{\gamma 1}\Big)\,\omega\,\sigma_f\Big\}\Big\}
\end{aligned}
$$

At time u, $\omega$ denotes the difference between the Wiener process at u and the Wiener process at u=0

**%13 /. ω → (W_u − W_0)**

$$\Bigg\{\Bigg\{\left(\frac{1}{2}e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma1}\,(u-u_0)}\right)tox_{f0}-\left(\frac{1}{2}e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma1}\,(u-u_0)}\right)\left(\frac{e^{-u\sqrt{\gamma1}}\left(1+e^{2u\sqrt{\gamma1}}\right)}{2\gamma1}-\frac{e^{-\sqrt{\gamma1}\,u_0}\left(1+e^{2\sqrt{\gamma1}\,u_0}\right)}{2\gamma1}\right)\gamma_0-$$

$$\left(\frac{e^{-\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}-\frac{e^{\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}\right)\left(\frac{e^{-u\sqrt{\gamma1}}\left(-1+e^{2u\sqrt{\gamma1}}\right)}{2\sqrt{\gamma1}}-\frac{e^{-\sqrt{\gamma1}\,u_0}\left(-1+e^{2\sqrt{\gamma1}\,u_0}\right)}{2\sqrt{\gamma1}}\right)\gamma_0+$$

$$\left(\frac{e^{-\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}-\frac{e^{\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}\right)\nu_{f0}+$$

$$\Bigg(\left(\frac{1}{2}e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma1}\,(u-u_0)}\right)\left(\frac{e^{-\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}-\frac{e^{\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}\right)+\left(\frac{1}{2}e^{-\sqrt{\gamma1}\,(u-u_0)}+\frac{1}{2}e^{\sqrt{\gamma1}\,(u-u_0)}\right)\left(-\frac{e^{-\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}+\frac{e^{\sqrt{\gamma1}\,(u-u_0)}}{2\sqrt{\gamma1}}\right)\Bigg)\sigma_f\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_u\right)+$$

$$\left(-\frac{1}{2}e^{-u\sqrt{\gamma1}}\left(1+e^{2u\sqrt{\gamma1}}\right)+\frac{1}{2}e^{-\sqrt{\gamma1}\,u_0}\left(1+e^{2\sqrt{\gamma1}\,u_0}\right)\right)$$

$$\left(\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}} - \frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right) \sigma_f \left(-\{\{\omega\}\}_0 + \{\{\omega\}\}_u\right) +$$

$$\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} + \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)}\right)$$

$$\left(-\frac{e^{-u\sqrt{\gamma 1}}\left(-1 + e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}} + \frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1 + e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)$$

$$\sigma_f \left(-\{\{\omega\}\}_0 + \{\{\omega\}\}_u\right)\Big\},$$

$$\Big\{\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} \sqrt{\gamma 1} - \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)} \sqrt{\gamma 1}\right) \mathrm{tox}_{f0} -$$

$$\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} + \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)}\right)$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}\left(-1 + e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}} - \frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1 + e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right) \gamma_0 -$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}\left(1 + e^{2u\sqrt{\gamma 1}}\right)}{2\gamma 1} - \frac{e^{-\sqrt{\gamma 1}\,u_0}\left(1 + e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\gamma 1}\right)$$

$$\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} \sqrt{\gamma 1} - \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)} \sqrt{\gamma 1}\right) \gamma_0 +$$

$$\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} + \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)}\right) \nu_{f0} +$$

$$\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} + \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)}\right)$$

$$\left(-\frac{1}{2} e^{-u\sqrt{\gamma 1}}\left(1 + e^{2u\sqrt{\gamma 1}}\right) + \frac{1}{2} e^{-\sqrt{\gamma 1}\,u_0}\left(1 + e^{2\sqrt{\gamma 1}\,u_0}\right)\right)$$

$$\sigma_f \left(-\{\{\omega\}\}_0 + \{\{\omega\}\}_u\right) +$$

$$\left(\left(\frac{1}{2} e^{-\sqrt{\gamma 1}\,(u-u_0)} + \frac{1}{2} e^{\sqrt{\gamma 1}\,(u-u_0)}\right)^2 +\right.$$

$$\left(-\frac{e^{-\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}+\frac{e^{\sqrt{\gamma 1}\,(u-u_0)}}{2\sqrt{\gamma 1}}\right)\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}\right)\Bigg)\,\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+$$

$$\left(-\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)}{2\sqrt{\gamma 1}}+\frac{e^{-\sqrt{\gamma 1}\,u_0}\left(-1+e^{2\sqrt{\gamma 1}\,u_0}\right)}{2\sqrt{\gamma 1}}\right)$$

$$\left(\frac{1}{2}\,e^{-\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}-\frac{1}{2}\,e^{\sqrt{\gamma 1}\,(u-u_0)}\sqrt{\gamma 1}\right)$$

$$\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\}\}$$

**%14 /. {$u_0 \to 0$}**

$$\left\{\left\{\left(\frac{1}{2}\,e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)tox_{f0}-\right.\right.$$

$$\left(\frac{1}{2}\,e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-\frac{1}{\gamma 1}+\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\,\gamma 1}\right)\gamma_0-$$

$$\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\gamma_0}{2\sqrt{\gamma 1}}+$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\nu_{f0}+$$

$$\left(\left(\frac{1}{2}\,e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)+\right.$$

$$\left.\left(\frac{1}{2}\,e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\right)\sigma_f$$

$$(-\{\{\omega\}\}_0 + \{\{\omega\}\}_u) + \left(1 - \frac{1}{2} e^{-u\sqrt{\gamma 1}} \left(1 + e^{2u\sqrt{\gamma 1}}\right)\right)$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}} - \frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right) \sigma_f \, (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u) -$$

$$\frac{1}{2\sqrt{\gamma 1}} \left(e^{-u\sqrt{\gamma 1}} \left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2}\right)\right.$$

$$\left.\left(-1 + e^{2u\sqrt{\gamma 1}}\right) \sigma_f \, (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u)\right)\Big\},$$

$$\Big\{\left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} \sqrt{\gamma 1} - \frac{1}{2} e^{u\sqrt{\gamma 1}} \sqrt{\gamma 1}\right) \mathrm{tox}_{f0} -$$

$$\left(-\frac{1}{\gamma 1} + \frac{e^{-u\sqrt{\gamma 1}} \left(1 + e^{2u\sqrt{\gamma 1}}\right)}{2\,\gamma 1}\right)$$

$$\left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} \sqrt{\gamma 1} - \frac{1}{2} e^{u\sqrt{\gamma 1}} \sqrt{\gamma 1}\right) \gamma_0 -$$

$$\frac{e^{-u\sqrt{\gamma 1}} \left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2}\right) \left(-1 + e^{2u\sqrt{\gamma 1}}\right) \gamma_0}{2\sqrt{\gamma 1}} +$$

$$\left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2}\right) \nu_{f0} + \left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2}\right)$$

$$\left(1 - \frac{1}{2} e^{-u\sqrt{\gamma 1}} \left(1 + e^{2u\sqrt{\gamma 1}}\right)\right) \sigma_f \, (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u) +$$

$$\left(\left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2}\right)^2 + \left(-\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}} + \frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\right.$$

$$\left.\left(\frac{1}{2} e^{-u\sqrt{\gamma 1}} \sqrt{\gamma 1} - \frac{1}{2} e^{u\sqrt{\gamma 1}} \sqrt{\gamma 1}\right)\right)$$

$$\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)-\frac{1}{2\sqrt{\gamma 1}}\left(e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}\sqrt{\gamma 1}-\frac{1}{2}e^{u\sqrt{\gamma 1}}\sqrt{\gamma 1}\right)\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\right)\}\}$$

Set the starting values for $tox_f$[u,te] and $\nu_f$[u,te] equal to the values the individual had when he/she stopped smoking

$$\{\{\mathbf{tox_f[u,te]}\},\{\nu_f[u,te]\}\}=\%15\;/.\;\Big\{tox_{f0}\to\Big\{\frac{1}{2\gamma 1}\left(e^{-te\sqrt{\gamma 1}}\left(\left(-1+e^{te\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2te\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[te]\Big\},$$

$$\nu_{f0}\to\Big\{\frac{1}{\sqrt{\gamma 1}}\left(-\mathrm{Sinh}\left[te\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\left(p\,\delta+\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu c}[te]\Big\}\Big\}$$

— *General::spell1 :*
*Possible spelling error: new symbol name "toxc" is similar to existing symbol "tox".*

— *General::spell1 :*
*Possible spelling error: new symbol name "νc" is similar to existing symbol "ν".*

$$\Big\{\Big\{\Big\{-\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-\frac{1}{\gamma 1}+\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\gamma 1}\right)\gamma_0-$$

$$\frac{e^{-u\sqrt{\gamma 1}}\left(-1+e^{2u\sqrt{\gamma 1}}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\gamma_0}{2\sqrt{\gamma 1}}+$$

$$\left(\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)+\right.$$

$$\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\right)\sigma_f$$

$$(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\left(1-\frac{1}{2}e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)\right)$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)-$$

$$\frac{1}{2\sqrt{\gamma 1}}\left(e^{-u\sqrt{\gamma 1}}\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-1+e^{2u\sqrt{\gamma 1}}\right)\right.$$

$$\left.\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\right)+\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)$$

$$\left(\frac{1}{2\gamma 1}\left(e^{-te\sqrt{\gamma 1}}\left(\left(-1+e^{te\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2\,te\sqrt{\gamma 1}}\right)\right.\right.\right.$$

$$\left.\left.\left.\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)+\epsilon_{toxc}[te]\right)+$$

$$\left(\frac{e^{-u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}-\frac{e^{u\sqrt{\gamma 1}}}{2\sqrt{\gamma 1}}\right)\left(\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[te\sqrt{\gamma 1}\right]\gamma_0+\right.\right.$$

$$\left.\sqrt{\gamma 1}\left(p\,\delta+\text{Cosh}\left[te\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right)+$$

$$\left.\epsilon_{\nu c}[te]\right)\}\},\ \left\{\left\{-\left(-\frac{1}{\gamma 1}+\frac{e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)}{2\gamma 1}\right)\right.\right.$$

$$\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}\sqrt{\gamma 1}-\frac{1}{2}e^{u\sqrt{\gamma 1}}\sqrt{\gamma 1}\right)\gamma_0-$$

$$\frac{e^{-u\sqrt{\gamma 1}}\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(-1+e^{2u\sqrt{\gamma 1}}\right)\gamma_0}{2\sqrt{\gamma 1}}+$$

$$\left(\frac{1}{2}e^{-u\sqrt{\gamma 1}}+\frac{e^{u\sqrt{\gamma 1}}}{2}\right)\left(1-\frac{1}{2}e^{-u\sqrt{\gamma 1}}\left(1+e^{2u\sqrt{\gamma 1}}\right)\right)$$

$$\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+$$

$$\left(\left(\left(\frac{1}{2}\,e^{-u\sqrt{\gamma1}}+\frac{e^{u\sqrt{\gamma1}}}{2}\right)^2+\left(-\frac{e^{-u\sqrt{\gamma1}}}{2\sqrt{\gamma1}}+\frac{e^{u\sqrt{\gamma1}}}{2\sqrt{\gamma1}}\right)\right.\right.$$

$$\left.\left(\frac{1}{2}\,e^{-u\sqrt{\gamma1}}\sqrt{\gamma1}-\frac{1}{2}\,e^{u\sqrt{\gamma1}}\sqrt{\gamma1}\right)\right)$$

$$\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)-\frac{1}{2\sqrt{\gamma1}}\left(e^{-u\sqrt{\gamma1}}\right.$$

$$\left(-1+e^{2u\sqrt{\gamma1}}\right)\left(\frac{1}{2}\,e^{-u\sqrt{\gamma1}}\sqrt{\gamma1}-\frac{1}{2}\,e^{u\sqrt{\gamma1}}\sqrt{\gamma1}\right)$$

$$\left.\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\right)+$$

$$\left(\frac{1}{2}\,e^{-u\sqrt{\gamma1}}\sqrt{\gamma1}-\frac{1}{2}\,e^{u\sqrt{\gamma1}}\sqrt{\gamma1}\right)$$

$$\left(\frac{1}{2\gamma1}\left(e^{-te\sqrt{\gamma1}}\left(\left(-1+e^{te\sqrt{\gamma1}}\right)^2\gamma_0+\right.\right.\right.$$

$$\left.\left.\left(-1+e^{2\,te\sqrt{\gamma1}}\right)\sqrt{\gamma1}\,(p\,\delta-\nu_{c0})\right)\right)+$$

$$\left.\epsilon_{toxc}[te]\right)+\left(\frac{1}{2}\,e^{-u\sqrt{\gamma1}}+\frac{e^{u\sqrt{\gamma1}}}{2}\right)$$

$$\left(\frac{1}{\sqrt{\gamma1}}\left(-\mathrm{Sinh}\left[te\sqrt{\gamma1}\right]\gamma_0+\sqrt{\gamma1}\left(p\,\delta+\right.\right.\right.$$

$$\left.\left.\left.\mathrm{Cosh}\left[te\sqrt{\gamma1}\right](-p\,\delta+\nu_{c0})\right)\right)+\epsilon_{\nu c}[te]\right)\}\}\}$$

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FullSimplify[%]
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$$\left\{\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-(te+u)\sqrt{\gamma 1}}\left(\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})+2\,e^{(te+u)\sqrt{\gamma 1}}\,\gamma 1\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]-2\,e^{(te+u)\sqrt{\gamma 1}}\sqrt{\gamma 1}\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]\left(p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\epsilon_{\nu c}[te]\right)\right)\right)\right\}\right\},$$

$$\left\{\left\{\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\,\text{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})-\gamma 1\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]+\sqrt{\gamma 1}\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]\left(p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\epsilon_{\nu c}[te]\right)\right)\right\}\right\}\right\}$$

The expected value of $tox_f[u, te]$ and $\nu_f[u,te]$ are found by setting all Wiener processes and random errors equal to zero.

$$\text{\% /. }\{(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\to 0,\ \epsilon_{toxc}[te]\to 0,\ \epsilon_{\nu c}[te]\to 0\}$$

$$\left\{\left\{\left\{\frac{1}{2\,\gamma 1}\left(e^{-(te+u)\sqrt{\gamma 1}}\left(-2\,e^{(te+u)\sqrt{\gamma 1}}\,p\sqrt{\gamma 1}\,\delta\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]+\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\right)\right)\right\}\right\},$$

$$\left\{\left\{\frac{1}{\sqrt{\gamma 1}}\left(p\sqrt{\gamma 1}\,\delta\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]-\text{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\,\text{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)\right\}\right\}\right\}$$

The Wiener process here is given by all of the terms, minus the expected values, and then setting the other two errors to zero

**(%17 - %18) /. { $\epsilon_{toxc}$[te] → 0, $\epsilon_{vc}$[te] → 0}**

$$\Bigg\{\Bigg\{\Bigg\{-\frac{1}{2\gamma1}\Big(e^{-(te+u)\sqrt{\gamma1}}\Big(-2\,e^{(te+u)\sqrt{\gamma1}}\,p\sqrt{\gamma1}\,\delta\,\text{Sinh}\big[u\sqrt{\gamma1}\big]+\left(-1+e^{(te+u)\sqrt{\gamma1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma1}}\right)\sqrt{\gamma1}\,(p\,\delta-\nu_{c0})\Big)\Big)+\frac{1}{2\gamma1}\Big(e^{-(te+u)\sqrt{\gamma1}}\Big(\left(-1+e^{(te+u)\sqrt{\gamma1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma1}}\right)\sqrt{\gamma1}\,(p\,\delta-\nu_{c0})-2\,e^{(te+u)\sqrt{\gamma1}}\sqrt{\gamma1}\,\text{Sinh}\big[u\sqrt{\gamma1}\big]\,(p\,\delta+\sigma_f(-\{\{\omega\}\}_0+\{\{\omega\}\}_u))\Big)\Big)\Bigg\}\Bigg\},$$

$$\Bigg\{\Bigg\{-\frac{1}{\sqrt{\gamma1}}\Big(p\sqrt{\gamma1}\,\delta\,\text{Cosh}\big[u\sqrt{\gamma1}\big]-\text{Sinh}\big[(te+u)\sqrt{\gamma1}\big]\gamma_0+\sqrt{\gamma1}\,\text{Cosh}\big[(te+u)\sqrt{\gamma1}\big](-p\,\delta+\nu_{c0})\Big)+\frac{1}{\sqrt{\gamma1}}\Big(-\text{Sinh}\big[(te+u)\sqrt{\gamma1}\big]\gamma_0+\sqrt{\gamma1}\,\text{Cosh}\big[(te+u)\sqrt{\gamma1}\big](-p\,\delta+\nu_{c0})+\sqrt{\gamma1}\,\text{Cosh}\big[u\sqrt{\gamma1}\big](p\,\delta+\sigma_f(-\{\{\omega\}\}_0+\{\{\omega\}\}_u))\Big)\Bigg\}\Bigg\}\Bigg\}$$

**FullSimplify[%19]**

$$\Bigg\{\Bigg\{\Bigg\{\frac{\text{Sinh}\big[u\sqrt{\gamma1}\big]\,\sigma_f\,(\{\{\omega\}\}_0-\{\{\omega\}\}_u)}{\sqrt{\gamma1}}\Bigg\}\Bigg\},\ \Big\{\Big\{\text{Cosh}\big[u\sqrt{\gamma1}\big]\,\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\Big\}\Big\}\Bigg\}$$

In the above statement of the Wiener process for $tox_f$[u, te], the Wiener process is going backward. Multiply this term by -1 and change the signs in the Wiener processes. Call these errors $\epsilon_{\text{toxf}}[u]$ and $\epsilon_{\gamma f}'[u]$, respectively.

$$\left\{\frac{-\text{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)}{\sqrt{\gamma 1}},\ \text{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\right\}$$

$$\left\{-\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)}{\sqrt{\gamma 1}},\ \text{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)\right\}$$

The coefficient on the $\epsilon_{\text{toxc}}[te]$error is found by taking the whole expression, subtracting off the expected values, the Wiener process, and setting $\epsilon_{\gamma c}[te] \to 0$

**(%17 - %18 - %21) /. $\epsilon_{vc}$[te] → 0**

$$\Big\{\Big\{\Big\{-\frac{1}{2\,\gamma 1}\Big(e^{-(te+u)\sqrt{\gamma 1}}\Big(-2\,e^{(te+u)\sqrt{\gamma 1}}\,p\,\sqrt{\gamma 1}\,\delta\,\mathrm{Sinh}\big[u\sqrt{\gamma 1}\big]+\big(-1+e^{(te+u)\sqrt{\gamma 1}}\big)^2\gamma_0+\big(-1+e^{2(te+u)\sqrt{\gamma 1}}\big)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\Big)\Big)+$$

$$\frac{\mathrm{Sinh}\big[u\sqrt{\gamma 1}\big]\,\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)}{\sqrt{\gamma 1}}+$$

$$\frac{1}{2\,\gamma 1}\Big(e^{-(te+u)\sqrt{\gamma 1}}\Big(\big(-1+e^{(te+u)\sqrt{\gamma 1}}\big)^2\gamma_0+\big(-1+e^{2(te+u)\sqrt{\gamma 1}}\big)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})-2\,e^{(te+u)\sqrt{\gamma 1}}\sqrt{\gamma 1}\,\mathrm{Sinh}\big[u\sqrt{\gamma 1}\big]\,(p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u))+2\,e^{(te+u)\sqrt{\gamma 1}}\,\gamma 1\,\mathrm{Cosh}\big[u\sqrt{\gamma 1}\big]\,\epsilon_{toxc}[te]\Big)\Big)\Big\}\Big\},$$

$$\Big\{\Big\{-\frac{1}{\sqrt{\gamma 1}}\Big(p\,\sqrt{\gamma 1}\,\delta\,\mathrm{Cosh}\big[u\sqrt{\gamma 1}\big]-\mathrm{Sinh}\big[(te+u)\sqrt{\gamma 1}\big]\,\gamma_0+\sqrt{\gamma 1}\,\mathrm{Cosh}\big[(te+u)\sqrt{\gamma 1}\big]\,(-p\,\delta+\nu_{c0})\Big)-$$

$$\mathrm{Cosh}\big[u\sqrt{\gamma 1}\big]\,\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+$$

$$\frac{1}{\sqrt{\gamma 1}}\Big(-\mathrm{Sinh}\big[(te+u)\sqrt{\gamma 1}\big]\,\gamma_0+\sqrt{\gamma 1}\,\mathrm{Cosh}\big[(te+u)\sqrt{\gamma 1}\big]\,(-p\,\delta+\nu_{c0})+\sqrt{\gamma 1}\,\mathrm{Cosh}\big[u\sqrt{\gamma 1}\big]\,(p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u))-\gamma 1\,\mathrm{Sinh}\big[u\sqrt{\gamma 1}\big]\,\epsilon_{toxc}[te]\Big)\Big\}\Big\}\Big\}$$

**FullSimplify[%22]**

$$\left\{\left\{\left\{\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\text{toxc}}[\text{te}]\right\}\right\},\left\{\left\{-\sqrt{\gamma 1}\ \text{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\text{toxc}}[\text{te}]\right\}\right\}\right\}$$

The coefficient on the $\epsilon_{\gamma c}[\text{te}]$ is found by taking the whole expression, subtracting off the expected values, the Wiener process, and setting $\epsilon_{\text{toxc}}[\text{te}] \to 0$

**(%17 - %18 - %21) /. $\epsilon_{toxc}$[te] → 0**

$$\left\{\left\{\left\{-\frac{1}{2\,\gamma 1}\left(e^{-(te+u)\sqrt{\gamma 1}}\left(-2\,e^{(te+u)\sqrt{\gamma 1}}\,p\sqrt{\gamma 1}\;\delta\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]+\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\;(p\,\delta-\nu_{c0})\right)\right)+\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)}{\sqrt{\gamma 1}}+\frac{1}{2\,\gamma 1}\left(e^{-(te+u)\sqrt{\gamma 1}}\left(\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\;(p\,\delta-\nu_{c0})-2\,e^{(te+u)\sqrt{\gamma 1}}\sqrt{\gamma 1}\;\text{Sinh}\left[u\sqrt{\gamma 1}\right](p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\epsilon_{\nu c}[te])\right)\right)\right\}\right\},$$

$$\left\{\left\{-\frac{1}{\sqrt{\gamma 1}}\left(p\sqrt{\gamma 1}\;\delta\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]-\text{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\;\text{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\right)-\text{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\frac{1}{\sqrt{\gamma 1}}\left(-\text{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\;\text{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})+\sqrt{\gamma 1}\;\text{Cosh}\left[u\sqrt{\gamma 1}\right](p\,\delta+\sigma_f\,(-\{\{\omega\}\}_0+\{\{\omega\}\}_u)+\epsilon_{\nu c}[te])\right)\right\}\right\}\right\}$$

**FullSimplify**[%]

$$\left\{\left\{\left\{-\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\nu c}[te]}{\sqrt{\gamma 1}}\right\}\right\},\left\{\left\{\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\nu c}[te]\right\}\right\}\right\}$$

This is a check. Do all the components add up to the whole?

**%17 – (%18 + %21 + %23 + %25)**

$$\Bigg\{\Bigg\{\Bigg\{-\frac{1}{2\gamma 1}\Big(e^{-(te+u)\sqrt{\gamma 1}}\Big(-2e^{(te+u)\sqrt{\gamma 1}}\,p\sqrt{\gamma 1}\,\delta\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]+\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})\Big)\Big)+$$

$$\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_u\right)}{\sqrt{\gamma 1}}-\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]+\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\nu c}[te]}{\sqrt{\gamma 1}}+$$

$$\frac{1}{2\gamma 1}\Big(e^{-(te+u)\sqrt{\gamma 1}}\Big(\left(-1+e^{(te+u)\sqrt{\gamma 1}}\right)^2\gamma_0+\left(-1+e^{2(te+u)\sqrt{\gamma 1}}\right)\sqrt{\gamma 1}\,(p\,\delta-\nu_{c0})+2e^{(te+u)\sqrt{\gamma 1}}\,\gamma 1\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]-2e^{(te+u)\sqrt{\gamma 1}}\sqrt{\gamma 1}\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]\left(p\,\delta+\sigma_f\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_u\right)+\epsilon_{\nu c}[te]\right)\Big)\Big)\Bigg\}\Bigg\},$$

$$\Bigg\{\Bigg\{-\frac{1}{\sqrt{\gamma 1}}\Big(p\sqrt{\gamma 1}\,\delta\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]-\text{Sinh}\left[(te+u)\sqrt{\gamma 1}\right]\gamma_0+\sqrt{\gamma 1}\,\text{Cosh}\left[(te+u)\sqrt{\gamma 1}\right](-p\,\delta+\nu_{c0})\Big)-\text{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\left(-\{\{\omega\}\}_0+\{\{\omega\}\}_u\right)+$$

$$\sqrt{\gamma 1}\ \mathrm{Sinh}\left[u \sqrt{\gamma 1}\right] \epsilon_{toxc}[te] - \mathrm{Cosh}\left[u \sqrt{\gamma 1}\right] \epsilon_{\nu c}[te] + \frac{1}{\sqrt{\gamma 1}} \Big(-\mathrm{Sinh}\left[(te+u) \sqrt{\gamma 1}\right] \gamma_0 + \sqrt{\gamma 1}\ \mathrm{Cosh}\left[(te+u)\sqrt{\gamma 1}\right] (-p\,\delta + \nu_{c0}) - \gamma 1\ \mathrm{Sinh}\left[u\sqrt{\gamma 1}\right] \epsilon_{toxc}[te] + \sqrt{\gamma 1}\ \mathrm{Cosh}\left[u \sqrt{\gamma 1}\right] (p\,\delta + \sigma_f (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u) + \epsilon_{\nu c}[te])\Big)\}\}\}$$

**FullSimplify[%26]**

{{{0}}, {{0}}}

We now derive the variance of the Wiener process. Start with the process

**%21**

$$\left\{-\frac{\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right] \sigma_f (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u)}{\sqrt{\gamma 1}},\ \mathrm{Cosh}\left[u\sqrt{\gamma 1}\right] \sigma_f (-\{\{\omega\}\}_0 + \{\{\omega\}\}_u)\right\}$$

The variance of the Wiener process is

$$\left\{\left(-\frac{\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]\sigma_f}{\sqrt{\gamma 1}}\right)\text{^}2\,u,\ \left(\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]\sigma_f\right)\text{^}2\,u\right\}$$

$$\left\{\frac{u\,\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2 \sigma_f^2}{\gamma 1},\ u\,\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2 \sigma_f^2\right\}$$

The whole error term that includes $\epsilon_{toxc}[te]$ is

**%23**

$$\left\{\left\{\left\{\mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]\right\}\right\},\ \left\{\left\{-\sqrt{\gamma 1}\ \mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{toxc}[te]\right\}\right\}\right\}$$

The variance of these terms are

$$\left\{\left(\mathbf{Cosh}\left[\mathbf{u}\sqrt{\gamma 1}\right]\right)\mathbf{\hat{}2\ V[\epsilon_{toxc}[te]]},\ \left(-\sqrt{\gamma 1}\ \mathbf{Sinh}\left[\mathbf{u}\sqrt{\gamma 1}\right]\right)\mathbf{\hat{}2\ V[\epsilon_{toxc}[te]]}\right\}$$

$$\left\{\text{Cosh}\left[u\sqrt{\gamma 1}\right]^2 V[\epsilon_{toxc}[te]],\ \gamma 1\ \text{Sinh}\left[u\sqrt{\gamma 1}\right]^2 V[\epsilon_{toxc}[te]]\right\}$$

Next is the expression of $V[\epsilon_{toxc}[te]$ from Appendix 1 :

$$\left\{-\frac{\mathbf{Sinh}\left[\mathbf{te}\sqrt{\gamma 1}\right]\sigma_c}{\sqrt{\gamma 1}}\right\}\mathbf{\hat{}2\ te}$$

$$\left\{\frac{te\ \text{Sinh}\left[te\sqrt{\gamma 1}\right]^2 \sigma_c^2}{\gamma 1}\right\}$$

Putting them together

$$\left\{\mathbf{Cosh}\left[\mathbf{u}\sqrt{\gamma 1}\right]^2\left(\frac{\mathbf{te\ Sinh}\left[\mathbf{te}\sqrt{\gamma 1}\right]^2\sigma_c^2}{\gamma 1}\right),\ \gamma 1\ \mathbf{Sinh}\left[\mathbf{u}\sqrt{\gamma 1}\right]^2\left(\frac{\mathbf{te\ Sinh}\left[\mathbf{te}\sqrt{\gamma 1}\right]^2\sigma_c^2}{\gamma 1}\right)\right\}$$

$$\left\{\frac{te\ \text{Cosh}\left[u\sqrt{\gamma 1}\right]^2\ \text{Sinh}\left[te\sqrt{\gamma 1}\right]^2\sigma_c^2}{\gamma 1},\ te\ \text{Sinh}\left[te\sqrt{\gamma 1}\right]^2\ \text{Sinh}\left[u\sqrt{\gamma 1}\right]^2\sigma_c^2\right\}$$

The $\epsilon_{\gamma c}[te]$ error term is

**%25**

$$\left\{\left\{\left\{-\frac{\text{Sinh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\gamma c}[te]}{\sqrt{\gamma 1}}\right\}\right\},\ \left\{\left\{\text{Cosh}\left[u\sqrt{\gamma 1}\right]\epsilon_{\gamma c}[te]\right\}\right\}\right\}$$

The variance of this term is

$$\left\{\left(-\frac{\mathbf{Sinh}\left[\mathbf{u}\sqrt{\gamma 1}\right]}{\sqrt{\gamma 1}}\right)\mathbf{\hat{}2\,V[\,\epsilon_{\nu c}[te]]},\right.$$

$$\left.\left(\mathbf{Cosh}\left[\mathbf{u}\sqrt{\gamma 1}\right]\right)\mathbf{\hat{}2\,V[\,\epsilon_{\nu c}[te]]}\right\}$$

$$\left\{\frac{\mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2 V[\epsilon_{\nu c}[te]]}{\gamma 1},\ \mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2 V[\epsilon_{\nu c}[te]]\right\}$$

Next is the expression of $V[\epsilon_{\nu c}[te]$ from TRDRP1.nb :

$$\left\{\mathbf{te\,Cosh}\left[\mathbf{te}\sqrt{\gamma 1}\right]^2 \sigma_c^2\right\}$$

$$\left\{te\,\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]^2 \sigma_c^2\right\}$$

Putting them together

$$\left\{\left(-\frac{\mathbf{Sinh}\left[\mathbf{u}\sqrt{\gamma 1}\right]}{\sqrt{\gamma 1}}\right)\mathbf{\hat{}2}\left\{\mathbf{te\,Cosh}\left[\mathbf{te}\sqrt{\gamma 1}\right]^2 \sigma_c^2\right\},\right.$$

$$\left.\left(\mathbf{Cosh}\left[\mathbf{u}\sqrt{\gamma 1}\right]\right)\mathbf{\hat{}2}\left\{\mathbf{te\,Cosh}\left[\mathbf{te}\sqrt{\gamma 1}\right]^2 \sigma_c^2\right\}\right\}$$

$$\left\{\left\{\frac{te\,\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]^2 \mathrm{Sinh}\left[u\sqrt{\gamma 1}\right]^2 \sigma_c^2}{\gamma 1}\right\},\right.$$

$$\left.\left\{te\,\mathrm{Cosh}\left[te\sqrt{\gamma 1}\right]^2 \mathrm{Cosh}\left[u\sqrt{\gamma 1}\right]^2 \sigma_c^2\right\}\right\}$$

The variance of {$tox_f$[te,u], $\nu_f$[te,u]} is equal to the sum of the variances of the three error terms in its expression (%29 + %33 + %37)

**%29 + %33 + %37**

$$\left\{\left\{\frac{\text{te}\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]^2\,\text{Sinh}\left[\text{te}\sqrt{\gamma 1}\right]^2\,\sigma_c^2}{\gamma 1}+\frac{\text{te}\,\text{Cosh}\left[\text{te}\sqrt{\gamma 1}\right]^2\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_c^2}{\gamma 1}+\frac{u\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_f^2}{\gamma 1}\right\},\left\{\text{te}\,\text{Cosh}\left[\text{te}\sqrt{\gamma 1}\right]^2\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_c^2+\text{te}\,\text{Sinh}\left[\text{te}\sqrt{\gamma 1}\right]^2\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_c^2+u\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_f^2\right\}\right\}$$

**FullSimplify**[%]

$$\left\{\left\{\frac{1}{\gamma 1}\left(\frac{1}{4}\,\text{te}\left(-2+\text{Cosh}\left[2\,(\text{te}-u)\sqrt{\gamma 1}\right]+\text{Cosh}\left[2\,(\text{te}+u)\sqrt{\gamma 1}\right]\right)\sigma_c^2+u\,\text{Sinh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_f^2\right)\right\},\left\{\frac{1}{4}\,\text{te}\left(2+\text{Cosh}\left[2\,(\text{te}-u)\sqrt{\gamma 1}\right]+\text{Cosh}\left[2\,(\text{te}+u)\sqrt{\gamma 1}\right]\right)\sigma_c^2+u\,\text{Cosh}\left[u\sqrt{\gamma 1}\right]^2\,\sigma_f^2\right\}\right\}$$